Q11 * Mathematik * Anwendung der Ableitungsregeln

1. Zeigen Sie, dass die angegebenen Funktionen den Definitionsbereich $D_f = R$ besitzen. Bestimmen Sie jeweils alle Hoch-, Tief- und Terrassenpunkte der Funktion.

   a) $f(x) = \sqrt{x^2 + 2x + 5}$ b) $f(x) = 2 \cdot \sin(0{,}5 \cdot x + \pi)$

   c) $f(x) = \dfrac{x+2}{\sqrt{x^2+2}}$ d) $f(x) = \dfrac{9 - x^2}{x^2 + 4}$

2. Bestimmen Sie jeweils den Definitionsbereich D_f und die Ableitung $f'(x)$. Ermitteln Sie dann geeignet den Wertebereich W_f von f.

   a) $f(x) = \sqrt{4x - x^2}$ b) $f(x) = \sqrt{x^2 - 4x + 8}$

   c) $f(x) = \dfrac{2x+1}{\sqrt{x^2+1}}$ d) $f(x) = \dfrac{5 \cdot \sqrt{2x+1}}{x^2 + 5}$

   e) $f(x) = \dfrac{2x+1}{x^2+2}$ f) $f(x) = \sqrt{2 - \cos(0{,}5 \cdot \pi \cdot x)}$

   g) $f(x) = \dfrac{x-4}{x^2+1}$ h) $f(x) = \dfrac{4 - 2x^2}{x^2+1}$

Q11 * Mathematik * Anwendung der Ableitungsregeln * Lösungen

1. a) $f(x) = \sqrt{x^2 + 2x + 5}$; $D_f = R$, denn $x^2 + 2x + 5 = x^2 + 2x + 4 + 1 = (x+2)^2 + 1 > 0$

$$f'(x) = \frac{2x+2}{2\cdot\sqrt{x^2+2x+5}} = \frac{x+1}{\sqrt{x^2+2x+5}} \text{ und } f'(x) = 0 \Leftrightarrow x_1 = -1 \text{ und}$$

$$f(-1) = \sqrt{1^2 - 2 + 5} = 2 \text{ ; wegen } \lim_{x\to\pm\infty} f(x) = \infty \text{ folgt TIP}(-1/2)$$

b) $f(x) = 2\cdot\sin(0{,}5\cdot x + \pi)$; $D_f = R$; $f'(x) = 2\cdot\cos(0{,}5\cdot x + \pi)\cdot 0{,}5 = \cos(0{,}5\cdot x + \pi)$

$$f'(x) = 0 \Leftrightarrow 0{,}5\cdot x_k + \pi = (2k+1)\cdot\frac{\pi}{2} \text{ mit } k \in Z \Leftrightarrow 0{,}5\cdot x_k = (2k-1)\cdot\frac{\pi}{2} \Leftrightarrow$$

$$x_k = (2k-1)\cdot\pi \text{ und } f(x_1) = f(\pi) = 2\cdot\sin(1{,}5\cdot\pi) = -2$$

also $\text{TIP}((2k-1)\cdot\pi / -2)$ für $k \in Z$ ungerade und $\text{HOP}((2k-1)\cdot\pi / 2)$ für $k \in Z$ gerade

c) $f(x) = \frac{x+2}{\sqrt{x^2+2}}$ $D_f = R$, denn $x^2 + 2 > 0$

$$f'(x) = \frac{\sqrt{x^2+2} - (x+2)\cdot\frac{1}{2}\cdot(x^2+2)^{-0{,}5}\cdot 2x}{x^2+2} = \frac{(x^2+2) - (x+2)\cdot x}{(x^2+2)\cdot\sqrt{x^2+2}} = \frac{2-2x}{(x^2+2)\cdot\sqrt{x^2+2}}$$

$f'(x) = 0 \Leftrightarrow 2 - 2x = 0 \Leftrightarrow x = 1$

wegen Vorzeichenwechsel von $f'(x)$ bei $x_1 = 1$ von + auf − also $\text{HOP}(1/\sqrt{3})$

d) $f(x) = \frac{9 - x^2}{x^2 + 4}$; $D_f = R$ wegen $x^2 + 4 > 0$

$$f'(x) = \frac{(x^2+4)\cdot(-2x) - (9-x^2)\cdot 2x}{(x^2+4)^2} = \frac{-2x^3 - 8x - 18x + 2x^3}{(x^2+4)^2} = \frac{-26x}{(x^2+4)^2}$$

$f'(x) = 0 \Leftrightarrow -26x = 0 \Leftrightarrow x_1 = 0$;

wegen Vorzeichenwechsel von $f'(x)$ bei x_1 von + auf − also $\text{HOP}(0/2{,}25)$

2. a) $f(x) = \sqrt{4x - x^2}$; $4x - x^2 \geq 0 \Leftrightarrow x\cdot(4-x) \geq 0 \Leftrightarrow x \in [0\,;\,4]$

$$f'(x) = \frac{4-2x}{2\cdot\sqrt{4x-x^2}} = \frac{2-x}{\sqrt{4x-x^2}} \text{ und } f'(x) = 0 \Leftrightarrow x_1 = 2 \text{ und } f(x_1) = 2$$

wegen $f(0) = f(4) = 0$ also $\text{HOP}(2/2)$ und $W_f = [0\,;\,2]$

da $(x-2)^2 + f(x)^2 = ... = 4 = 2^2$ gilt, handelt es sich bei G_f um einen Halbkreis mit $M(2/0)$ und Radius $r = 2$.

b) $f(x) = \sqrt{x^2 - 4x + 8}$; wegen $x^2 - 4x + 8 = (x-2)^2 + 4 > 0$ gilt $D_f = R$

$$f'(x) = \frac{2x-4}{2\cdot\sqrt{x^2-4x+8}} = \frac{x-2}{\sqrt{x^2-4x+8}} \text{ und } f'(x) = 0 \Leftrightarrow x_1 = 2$$

wegen Vorzeichenwechsel von $f'(x)$ bei x_1 von − auf + also $\text{TIP}(2/2)$

wegen $\lim_{x\to\pm\infty} f(x) = \infty$ damit $W_f = [2;\infty[$

c) $f(x)=\frac{2x+1}{\sqrt{x^2+1}}$; $D_f=\mathbb{R}$ wegen $x^2+1>0$

$$f'(x)=\frac{\sqrt{x^2+1}\cdot 2-(2x+1)\cdot\frac{2x}{2\cdot\sqrt{x^2+1}}}{(x^2+1)}=\frac{(x^2+1)\cdot 2-(2x+1)\cdot x}{(x^2+1)\cdot\sqrt{x^2+1}}=\frac{2-x}{(x^2+1)\cdot\sqrt{x^2+1}}$$

$f'(x)=0 \Leftrightarrow x_1=2$ und wegen Vorzeichenwechsel von $f'(x)$ bei x_1 von + auf − also $HOP(2/\sqrt{5})$; $\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\frac{2x}{\sqrt{x^2}}=\lim\limits_{x\to\pm\infty}\frac{2x}{|x|}=\pm 2$ also $W_f=]-2\,;\sqrt{5}\,]$

d) $f(x)=\frac{5\cdot\sqrt{2x+1}}{x^2+5}$; $D_f=[-\frac{1}{2};\infty[$, da $2x+1\geq 0 \Leftrightarrow x\geq -\frac{1}{2}$

$$f'(x)=5\cdot\frac{(x^2+5)\cdot\frac{2}{2\cdot\sqrt{2x+1}}-\sqrt{2x+1}\cdot 2x}{(x^2+5)^2}=5\cdot\frac{(x^2+5)-(2x+1)\cdot 2x}{(x^2+5)^2\cdot\sqrt{2x+1}}=$$

$$5\cdot\frac{x^2+5-4x^2-2x}{(x^2+5)^2\cdot\sqrt{2x+1}}=5\cdot\frac{-3x^2-2x+5}{(x^2+5)^2\cdot\sqrt{2x+1}}=$$

$f'(x)=0 \Leftrightarrow 3x^2+2x-5=0 \Leftrightarrow x_{1/2}=\frac{1}{6}\cdot(-2\pm\sqrt{4+4\cdot 3\cdot 5})=\frac{1}{3}\cdot(-1\pm 4) \Leftrightarrow$

$x_1=1$; $(x_2=-\frac{5}{3}\notin D_f)$; wegen $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{5\cdot\sqrt{2x}}{x^2}=\lim\limits_{x\to\infty}\frac{5\cdot\sqrt{2}}{x^{1,5}}=0$

und $f(-0,5)=0$ und $f(1)=\frac{5\cdot\sqrt{3}}{6}\approx 1,44$ also $HOP(1/\frac{5\cdot\sqrt{3}}{6})$ und $W_f=[0\,;\frac{5\cdot\sqrt{3}}{6}]$

e) $f(x)=\frac{2x+1}{x^2+2}$; $D_f=\mathbb{R}$ wegen $x^2+2>0$

$$f'(x)=\frac{(x^2+2)\cdot 2-(2x+1)\cdot 2x}{(x^2+2)^2}=\frac{2x^2+4-4x^2-2x}{(x^2+2)^2}=\frac{-2(x^2+x-2)}{(x^2+2)^2}$$

$f'(x)=0 \Leftrightarrow x^2+x-2=0 \Leftrightarrow (x-1)\cdot(x+2)=0 \Leftrightarrow x_1=1$; $x_2=-2$

wegen Vorzeichenwechsel von $f'(x)$ bei x_1 und x_2 $HOP(1/1)$ und $TIP(-2/-0,5)$

und daher $W_f=[-0,5\,;1\,]$

f) $f(x)=\sqrt{2-\cos(0,5\cdot\pi\cdot x)}$; $D_f=\mathbb{R}$ wegen $1\leq 2-\cos(0,5\cdot\pi\cdot x)\leq 3$

da $g(x)=\sqrt{x}$ eine streng monotone Funktion ist, folgt damit $W_f=[\sqrt{1}\,;\sqrt{3}\,]=[\,1\,;\sqrt{3}\,]$

$f'(x)=\frac{+\sin(0,5\cdot\pi\cdot x)\cdot 0,5\cdot\pi}{2\cdot\sqrt{2-\cos(0,5\cdot\pi\cdot x)}}=\frac{\pi\cdot\sin(0,5\cdot\pi\cdot x)}{4\cdot\sqrt{2-\cos(0,5\cdot\pi\cdot x)}}$ ist damit unnötig!

g) $f(x)=\frac{x-4}{x^2+1}$; $D_f=\mathbb{R}$ und $f'(x)=\frac{(x^2+1)\cdot 1-(x-4)\cdot 2x}{(x^2+1)^2}=\frac{-x^2+8x+1}{(x^2+1)^2}$

$f'(x)=0 \Leftrightarrow -x^2+8x+1=0 \Leftrightarrow x_{1/2}=\frac{1}{-2}\cdot(-8\pm\sqrt{64+4}) \Leftrightarrow$

$x_1=4-\sqrt{17}\approx -0,12$; $x_2=4+\sqrt{17}\approx 8,12$ und wegen $TIP(x_1/f(x_1))$, $HOP(x_2/f(x_2))$

daher $W_f=[f(x_1)\,;f(x_2)]=...=[-\frac{4+\sqrt{17}}{2}\,;\frac{-4+\sqrt{17}}{2}\,]\approx[-4,06\,;0,06\,]$

h) $f(x) = \dfrac{4-2x^2}{x^2+1}$; $D_f = \mathbb{R}$ wegen $x^2+1>0$ und

$$f'(x) = \frac{(x^2+1)\cdot(-4x)-(4-2x^2)\cdot 2x}{(x^2+1)^2} = \frac{-4x^3-4x-8x+4x^3}{(x^2+1)^2} = \frac{-12x}{(x^2+1)^2}$$

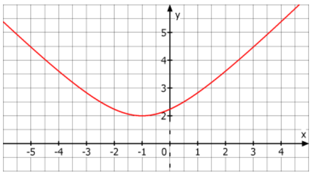
$f'(x)=0 \iff x_1 = 0$ und wegen Vorzeichenwechsel von $f'(x)$ bei x_1 von + auf −

also HOP(0/4) ; $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{-2x^2}{x^2} = \lim\limits_{x\to\pm\infty} -2 = -2$ und $f(x) \neq 2$ für alle $x \in \mathbb{R}$,

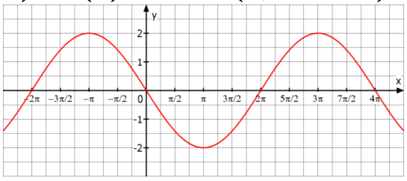
daher $W_f =]-2; 4]$

Bilder der Graphen

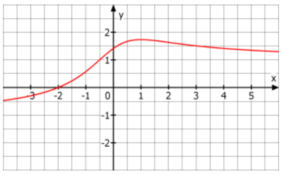
1. a) $f(x) = \sqrt{x^2+2x+5}$



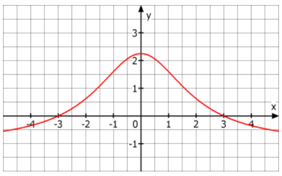
b) $f(x) = 2\cdot\sin(0{,}5\cdot x+\pi)$



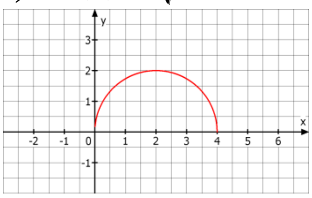
c) $f(x) = \dfrac{x+2}{\sqrt{x^2+2}}$



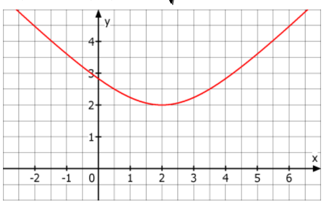
d) $f(x) = \dfrac{9-x^2}{x^2+4}$



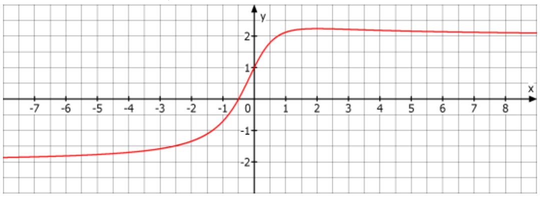
2. a) $f(x) = \sqrt{4x-x^2}$



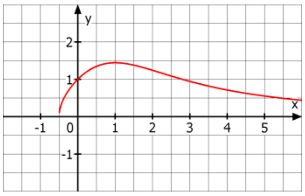
b) $f(x) = \sqrt{x^2-4x+8}$



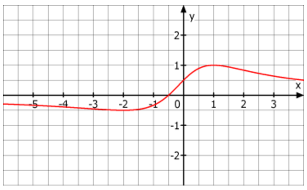
c) $f(x) = \dfrac{2x+1}{\sqrt{x^2+1}}$



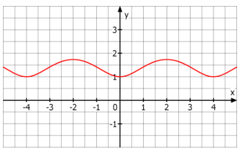
d) $f(x) = \dfrac{5\cdot\sqrt{2x+1}}{x^2+5}$



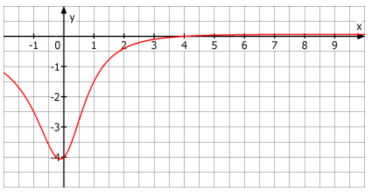
e) $f(x) = \dfrac{2x+1}{x^2+2}$



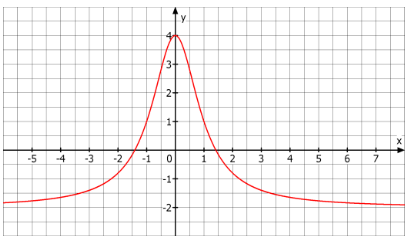
f) $f(x) = \sqrt{2-\cos(0{,}5\cdot\pi\cdot x)}$



g) $f(x) = \dfrac{x-4}{x^2+1}$

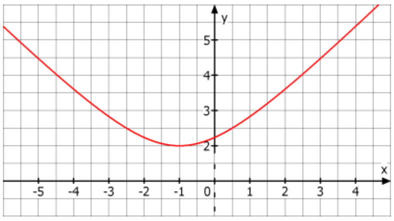


h) $f(x) = \dfrac{4-2x^2}{x^2+1}$

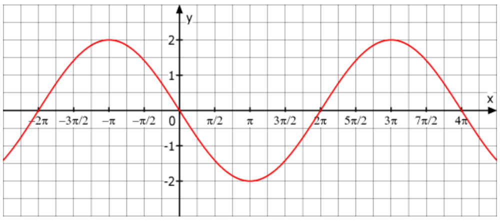


Q11 * Mathematik * Anwendung der Ableitungsregeln * Bilder der Graphen

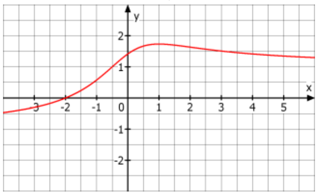
1\. a) $f(x) = \sqrt{x^2 + 2x + 5}$



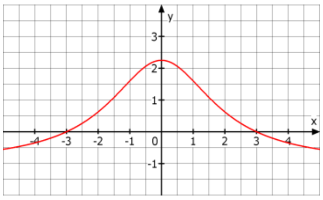
b) $f(x) = 2 \cdot \sin(0{,}5 \cdot x + \pi)$



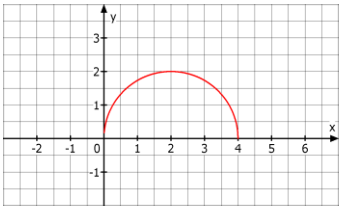
c) $f(x) = \dfrac{x + 2}{\sqrt{x^2 + 2}}$



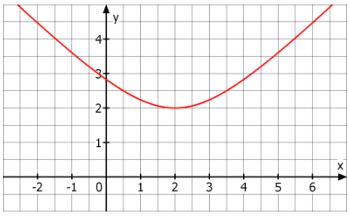
d) $f(x) = \dfrac{9 - x^2}{x^2 + 4}$



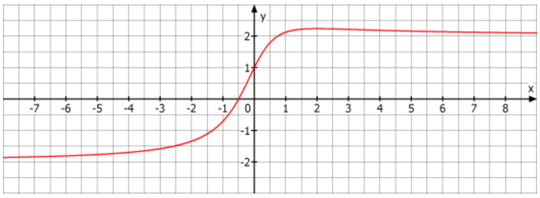
2\. a) $f(x) = \sqrt{4x - x^2}$



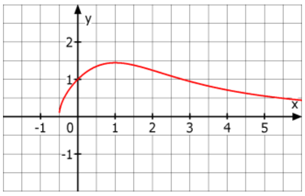
b) $f(x) = \sqrt{x^2 - 4x + 8}$



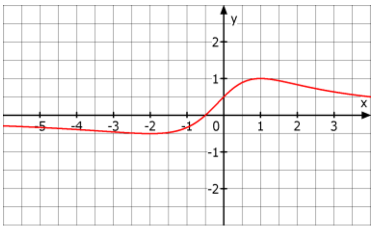
c) $f(x) = \dfrac{2x + 1}{\sqrt{x^2 + 1}}$



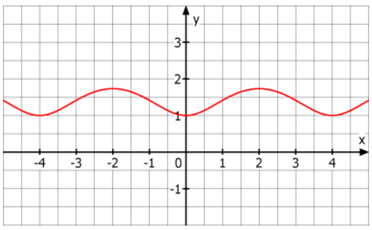
d) $f(x) = \dfrac{5 \cdot \sqrt{2x + 1}}{x^2 + 5}$



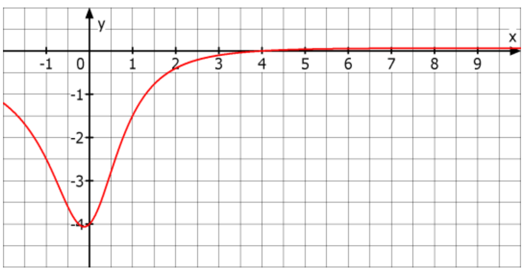
e) $f(x) = \dfrac{2x + 1}{x^2 + 2}$



f) $f(x) = \sqrt{2 - \cos(0{,}5 \cdot \pi \cdot x)}$



g) $f(x) = \dfrac{x - 4}{x^2 + 1}$



h) $f(x) = \dfrac{4 - 2x^2}{x^2 + 1}$